A GEOMETRICAL DETERMINATION OF THE CANONICAL QUADRIC OF WILCZYNSKI

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In the study of the projective differential properties of curved surfaces a canonical development for the equation of the surface and a geometrical determination of the associated tetrahedron of reference are of fundamental importance. Wilczynski[1] was the first to solve this problem. The vertices of his canonical tetrahedron can be easily located as soon as a certain quadric surface Q upon which they lie is determined. Green[2] and others[3] have obtained similar canonical developments which take their simplest forms when the vertices of the associated tetrahedrons lie upon Q.

The quadric Q is commonly called the *canonical quadric of Wilczynski.* It was located by Wilczynski by means of a unodal cubic surface osculating the curved surface. The introduction of this canonical cubic surface complicates the situation considerably, especially since its determination before Q is known is a rather involved process. Recently Bompiani[4] obtained Q—apparently without recognizing it—by a geometrical process somewhat simpler than that of Wilczynski but little related to the problem of determining a canonical tetrahedron.

In the present paper the canonical quadric of Wilczynski is defined geometrically in an exceedingly simple manner by means of the axis of Cech, a line which is covariantly related to the curved surface and which is easily located geometrically.

Since most of the preliminary facts used in this paper are found in the memoir by Green,[2] we shall use his notation in order to facilitate reference. Accordingly, we refer our curved surface S to its asymptotic net and take the associated system of differential equations in the form

$$\begin{aligned} y_{uu} + 2by_v + fy &= 0, \\ y_{vv} + 2a'y_u + gy &= 0. \end{aligned} \tag{1}$$

Let the two asymptotic curves passing through a point y of the surface be denoted by C_u and C_v. The tangents at points of C_v to the curves v = const. generate a non-developable ruled surface $R^{(u)}$ and the tangents at points of C_u to the curves u = const. generate a similar surface $R^{(v)}$. The points

$$\rho = y_u - \beta y, \quad \sigma = y_v - \alpha y, \tag{2}$$

where α and β are functions of u and v, lie on the tangents to C_u and C_v, respectively, and the line l determined by them lies in the plane tangent to S at y. The tangent planes to $R^{(u)}$ at ρ and to $R^{(v)}$ at σ intersect in a line l' which passes through y and the point $y_{uv} - \alpha y_u - \beta y_v$. Green calls either of the lines l, l' the *reciprocal* of the other.

If $\beta = a'_u/2a'$, $\alpha = b_v/2b$, the lines l and l' are the directrices of Wilczynski of the first and second kind, respectively, and if $\beta = -b_u/4b$, $\alpha = -a'_v/4a'$, they are the canonical edges of Green of the first and second kind, respectively. These two pairs of lines have been located geometrically in several different ways. For example, in the former case the point ρ is the harmonic conjugate[5] of y with respect to the points where the flecnode curves of $R^{(u)}$ intersect $y\rho$, and in the latter case the point ρ is the pole of the tangent at y to C_v with respect to the osculating conic[6] of C_u at y. The corresponding points σ may be located in a similar manner.

The directrix of the first kind of Wilczynski and the canonical edge of the first kind of Green intersect in a point called the *canonical point* and their reciprocals lie in a plane called the *canonical plane*. The intersection of the canonical plane and the tangent plane to S at y is called the *canonical line*.

The harmonic conjugate of the directrix of the second kind of Wilczynski with respect to the canonical line and the canonical edge of the second kind of Green is the *projective normal* of Fubini, the pseudo-normal of Green. For it $\beta = -{}^1/_2(b_u/b + a'_u/a')$, $\alpha = -{}^1/_2(a'_v/a' + b_v/b)$. Again, the harmonic conjugate of the projective normal with respect to the directrix of Wilczynski and the canonical edge of Green, both of the second kind, is the axis of Cech. It is given by $\beta = -{}^1/_6(b_u/b - a'_u/a')$, $\alpha = -{}^1/_6(a'_v/a' - b_v/b)$. Both the projective normal and the axis of Cech have other geometrical definitions.

Green obtained a general expression representing a group of canonical developments, including that obtained by Wilczynski by choosing as the vertices of the canonical tetrahedron the four points

$$y, \rho = y_u - \beta y, \sigma = y_v - \alpha y, \tau = y_{uv} - \alpha y_u - \beta y_v + \alpha\beta y, \tag{3}$$

where α and β are functions of u and v which need only be properly assigned in order to obtain any one of the several developments. Since any point X in space is defined by an expression of the form $x_1 y + x_2\rho + x_3\sigma + x_4\tau$, the coördinates of X may be taken to be (x_1, x_2, x_3, x_4). In this new coördinate system the general expression for the several canonical developments has the form

$$\begin{aligned}\zeta = \xi\eta + {}^2/_3 b\xi^3 + {}^2/_3 a'\eta^3 + {}^1/_6(4b\beta + b_u)\xi^4 + {}^2/_3(b_v - 2b\alpha)\xi^3\eta \\ + {}^2/_3(a'_u - 2a'\beta)\xi\eta^3 + {}^1/_6(4a'\alpha + a'_v)\eta^4 + \ldots,\end{aligned} \tag{4}$$

where $\xi = x_2/x_1$, $\eta = x_3/x_1$, $\zeta = x_4/x_1$. Cutting off the development after the first term we have